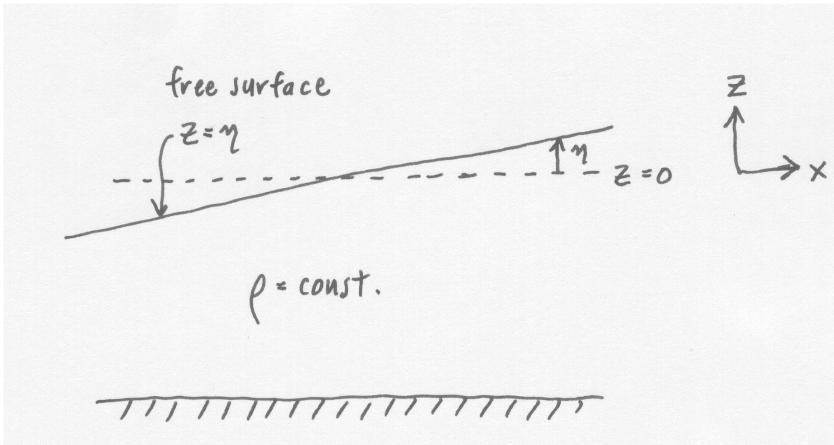
GDF I, Solutions to Problem set #1

1) The physical problem looks like:



(a) The horizontal pressure gradient is most easily found by first integrating the hydrostatic equation $p_z = -\rho g$ from some arbitrary position z to the free surface at $z = \eta$. This gives $p_{atm} - p = -\rho g(\eta - z)$. Then taking $\partial/\partial x$ we find the desired expression:

[*] $$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x}$$

(b) Using [*] the x-momentum equation may be written as $u_t = -g\eta_x$. Integrating over some time this gives the solution:

$$\Delta u = -g\eta_x \Delta t = -\left(9.8 \text{ m s}^{-2}\right)\left(10^{-5}\right)\left(86400 \text{ s}\right) = -8.5 \text{ m s}^{-1}$$

(c) In the upper left corner the 500 mb surface has strongest gradients in the y-direction. The gradient is about:

$$\left.\frac{\partial z}{\partial y}\right|_p = \frac{-360 \text{ m}}{10° \text{ latitude}} \times \frac{1° \text{ latitude}}{111.3 \times 10^3 \text{ m}} = -3.2 \times 10^{-4}$$

And so the velocity change after a day is given by:

$$\Delta v = -\frac{1}{\rho}\frac{\partial p}{\partial y}\Delta t = -\left.\frac{\partial \Phi}{\partial y}\right|_p \Delta t = -g\left.\frac{\partial z}{\partial y}\right|_p \Delta t$$

$$= -9.8\frac{\text{m}}{\text{s}^2}\times\left(-3.2\times10^{-4}\right)\times\left(8.64\times10^4\ \text{s}\right) = 271\ \text{m s}^{-1}$$

which is pretty fast!

2) From [I] and [II] from problem 3 it is easy to show (as done in class) that for isentropic compression of an ideal gas:

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \quad \text{and} \quad \frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

Thus when we compress a cylinder of gas to half its original volume the three state variables change in the ratios:

$$\frac{\rho_2}{\rho_1} = 2 \qquad \frac{p_2}{p_1} = 2^{1.4} = 2.6 \qquad \frac{T_2}{T_1} = 2^{0.4} = 1.3$$

Why would this be? Thinking about the molecular origins of pressure, we would expect p <u>to at least double</u>, simply because the density has doubled, and so twice as many molecules are hitting the walls of the volume per unit time. You might object that the wall area has decreased, but this does not affect pressure, which is defined *per unit area*. The fact that the pressure *more* than doubles is because in the course of the compression the moving wall has imparted kinetic energy to the molecules hitting it, thus increasing the temperature. This in turn becomes a second means of increasing the pressure.

3a) From expressions [I] and [II] one may show directly that $p^{1-\gamma}(RT)^{\gamma} = \text{constant}$.

Then taking the natural log we find: $(1-\gamma)\ln p + \gamma \ln R + \gamma \ln T = \text{constant}$ (note that it is a different constant now!). Taking $\partial/\partial z$ we find: $(1-\gamma)\frac{\partial p}{\partial z}\frac{1}{p} + \gamma \frac{\partial T}{\partial z}\frac{1}{T} = 0$. We use the hydrostatic balance [III] to substitute for the vertical pressure gradient in this, and make use of the ideal gas law [I] to find the answer:

$$\frac{\partial T}{\partial z} = \left(\frac{1-\gamma}{\gamma}\right)\frac{\rho g T}{p} = \frac{(1-\gamma)g}{\gamma R} = -\frac{g}{C_p} \cong -10°\text{C km}^{-1}$$

Thus the temperature decreases linearly with height. Note that this solution <u>requires that the whole atmosphere be isentropic</u>. In this case you can move a parcel to any height and it will always have the same temperature and pressure as the surrounding air. This is <u>not</u> true for isentropic parcel movement in a more general stratification (e.g. the US Standard Atmosphere, Vallis Fig. 12.22).

(b) From the ideal gas law [I] and the hydrostatic balance [III] you can show:

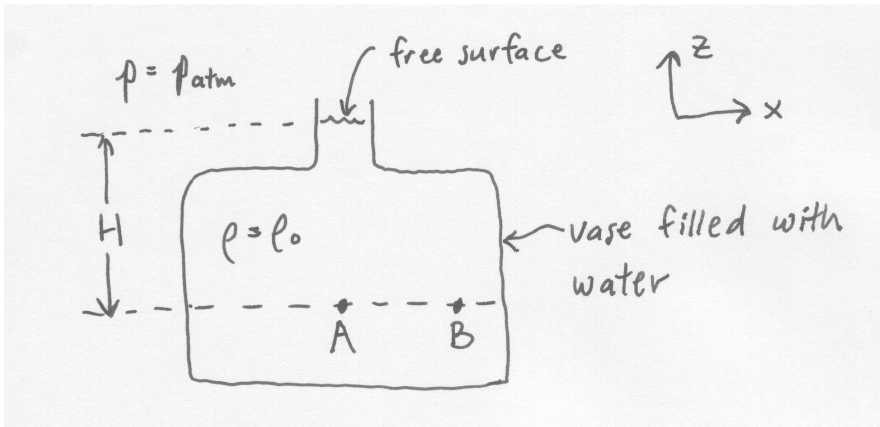
$p_z = -\frac{g}{RT}p$. For the case of constant temperature this is a simple first-order ODE, which has solution:

$$p = p_0 \exp\left(-\frac{g}{RT}z\right)$$

Thus the e-folding height is given by:

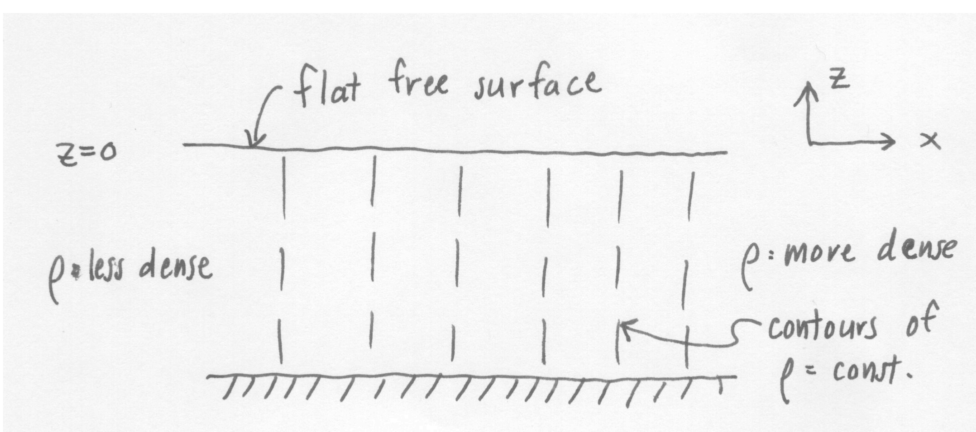
$$\text{Scale Height} = \frac{\text{RT}}{\text{g}} = \frac{287\text{ J}}{\text{kg K}} \times (250\text{ K}) \times \frac{\text{s}^2}{9.8\text{ m}} = 7.3\text{ km}$$

4) The physical situation for this problem looks like



The pressure at A and B is the same, and is given by $p = p_{atm} + \rho_0 g H$. This should be obvious, but it can be hard to prove from the hydrostatic relation because you have to account for the pressure on the wall of the vase above point B. Another approach would be to appeal to the x-momentum equation and assert that since there is no horizontal acceleration $p_x = 0$ and so the two pressures must be identical.

5) Here the physical situation looks like:

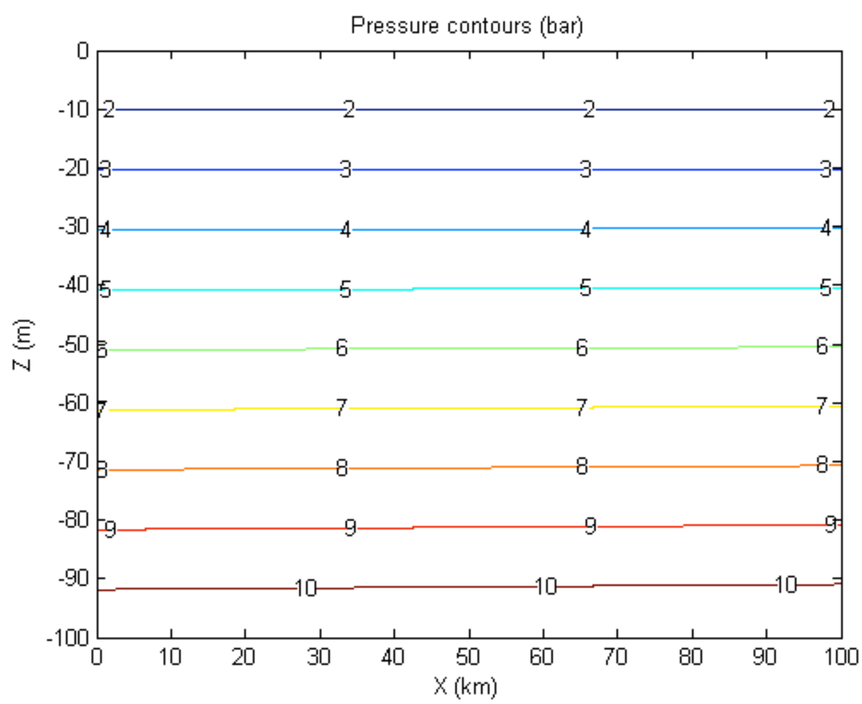


You can (without loss of generality) create an expression for the density of the form:

$\rho = \rho_0 + \frac{\partial \rho}{\partial x} x$. Then use this in the hydrostatic balance and integrate from some depth z to the surface at $z = 0$ to show that

$$p = p_{atm} - g\left(\rho_0 + \frac{\partial \rho}{\partial x} x\right) z .$$

It is interesting to plot this using MATLAB in order to get a graphical feel for how the pressure field might look under different choices of parameters. With a density gradient of 1 kg over 10 km, and 100 m water depth, the field looks like:



The code I used here was:

```
% p_contours.m  1/12/2006  Parker MacCready
%
% This plots pressure contours for a vertically well-mixed fluid that has
% some drho/dx
%
% Written for the GFD I first problem set

clear
x = [0:100:1e5];     % 100 km domain [m]
z = [-100:10:0];     % 100 m deep [m]
g = 9.8;             % gravity [m s-2]
```

```
rho0 = 1000;         % background density [kg m-3]
drho_dx = 1/10e3;    % drho/dx = 1 kg m-3 in 10 km [kg m-4]
p_atm = 1e5;         % atmospheric pressure [Pa]

% make complete coordinates
[X,Z] = meshgrid(x,z);
% calculate the pressure from the analytical solution
p = p_atm - g*(rho0 + drho_dx*X).*Z;

figure
[cc,hh] = contour(X/1000,Z,p/1e5);
clabel(cc,hh);
xlabel('X (km)');
ylabel('Z (m)');
title('Pressure contours (bar)')
```

It is surprising how much the actual pressure contours are dominated by the vertical gradient (which causes no motion at all) compared with the relatively tiny horizontal gradient. The horizontal acceleration field is governed by

$$\frac{\partial u}{\partial t} = \frac{g}{\rho_0}\frac{\partial \rho}{\partial x} z$$

Where I have approximated the density in the denominator by the background density ρ_0. Thus the acceleration is independent of *x*, is zero at the surface, and increases linearly with depth. Note that the acceleration is to the <u>left</u> if density increases to the <u>right</u>.